# U T I L I T Y   T H E O R Y

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These lectures will deal with utility theory, viewed in psychological perspective. Motivation has had little formal mathematical treatment in psychology proper; most of the mathematical work has been done in economics and statistics.

Consider a situation in which a person has the opportunity to accept or reject a money gamble. A possible first assumption is that the decision criterion used is the expected money return : accept a gamble if its expected return is positive. But no one behaves in this way. This is shown, for example, by the St. Petersburg Paradox. Suppose a fair coin is tossed until a head appears (on trial n) and the gambler is then paid $2^n$ dollars. How much would a person pay to play this game once ? The expected money return is infinite, so that this criterion would dictate that any sum, no matter how large, would be paid. Yet everyone has a limit above which he would not pay to play. Bernoulli accounted for this fact by suggesting that the utility of money increases with money, but at a decreasing rate, and that expected _utility_, rather than expected money return, is the criterion.

## 1. The Von Neumann-Morgenstern Theory of Utility

Economists have long sought to attach such utilities to goods - numbers that would allow goods to be treated in terms of classical economics. Traditional numerical representations in economics were unique only up to a monotonic transformation : they contained only ordinal information. Utility theory was advanced with the 1947 edition of von Neumann and Morgenstern's _Theory of Games_. Instead of considering choices between pure alternatives of commodities or of quantities of money, as was done traditionally, they considered choices between gambles, and utilized the probabilities involved in the gambles to further specify the utility representation. The main intuitive idea was that utility was something whose expected value people maximized, and the aim was to find plausible assumptions about humans that would lead them to behave this way, in terms of some appropriate utility function. (This approach is to be contrasted with one where a set of observed properties of behavior is the starting point). Psychologists have later asked whether the assumptions hold, i.e. whether people do in fact behave in terms of the theory.

1.1 Outline of the Theory

We have :

(1) a Boolean Algebra, E, of events and a probability measure, P, over E. E is dense in the sense that : given $\pi \in [0,1]$, $\exists e \in E \ni P(e) = \pi$.

(2) From E and an implicit set of pure alternatives we compose a set of gambles (plus pure alternatives) G : if $a, b \in G$ and $e \in E$ then $aeb \in G$. (" $aeb$ " is interpreted : pure alternative $a$ if $e$ occurs; alternative $b$ if e does not occur).

(3) There is a binary ordering relation, $\gtrsim$ , over the elements of G, representing preference or indifference. ( $>$ represents preference, $\sim$ represents indifference).

Now we want to find a set of axioms about the preference relation, such that an appropriate structure is imposed on G ; in particular we want to be able to derive the following representation theorem :

Representation Theorem : Function $u: G \rightarrow$ reals, such that

(i) $a \gtrsim b \Longleftrightarrow u(a) \geq u(b)$

(ii) $u(aeb) = u(a)p(e) + u(b)[I - p(e)]$ .

Any utility function must have property (i), i.e. must preserve the ordering of preferences. Property (ii) is the expected utility property. Before giving a set of axioms, we state two preliminary results.

Definition : A binary relation R on a set G is a weak ordering if for $a,b,c,\epsilon G$ , $aRa$ (reflexive), if $aRb$ and $bRc$ then $aRc$ (transitive), and for any $a,b$ either $aRb$ or $bRa$ (connected).

Theorem 1 : If the representation theorem holds, then $\succsim$ is a weak ordering.

Proof. Follows immediately from (i).

Theorem 2 : If u, $u^*$ are two representations satisfying the theorem, then $\exists K>0, C \ni u=Ku^*+C$ i.e. the representation is unique expect for its zero and unit, giving an interval scale.

Proof. Let $a,b \in G$, $a>b$, $u(a) = A$, $u(b)= B$, $u^*(a)=A^*$, $u^*(b)=B^*$, $K=\frac{A-B}{A^*-B^*}$ , $C= \frac{BA^* -AB^*}{A^* -B^*}$ . Note that $K>0$ by $a>b$ and (i).

Consider any $x \epsilon G$ . We wish to show $u(x) = Ku^*(x)+C$. There are five cases : we consider the one where $x>a>b$. By (i),$u(x)>u(a)>u(b)$ . Thus $\exists \pi \in [0,1] \ni u(a) = \pi u(x) + (I-\pi)u(b)$ . By density of $E, e \in E \ni P(e) = \pi$. Hence $u(a) = P(e)u(x)+[I-P(e)]u(b)$.By (II), $u(a) = u(xeb)$, and so $a \sim xeb$ . By the same argument $u^*(a) = u^*(xeb)$, and $u^*(a) = P(e)u^*(x)+ [i-P(e)] u^*(b)$ . Solving this equation and its parallel equation for P(e)and equating gives

$$\frac{A - B}{u(x)-B} = \frac{A^* - B^*}{u^*(x)-B^*}$$

Solving for u(x) gives the desired result.

To determine what conditions (axioms about $\succsim$ ) will make possible such a representation, the method is to consider what properties the representation theorem implies about $\succsim$ , and choose from this set a subset, reasonably minimal and plausible. We shall give one such possible set of axioms, for the special case when G has a maximum and minimum element. The restriction is not severe for practical cases, and generalization is easy. Our axioms, then, are

A1. $\succsim$ is a weak ordering. (Theorem 1) (Even this may not be true of humans).

A2. G : $\exists a, b \in G \ni$ for all $x \in G$, $a \succsim x \succsim b$.
(Max. and Min. elements)

A3. If $x \succsim y \succsim z$, then $\exists \alpha \in E \ni x \alpha z \sim y$.
(continuity assumption) (This may not be reasonable where one outcome is incommensurate with others, e.g. let x = \$2., y = \$1., z = death).

A4. If $x \sim y$ and $\alpha \in E$ then $x \alpha z \sim y \alpha z$ and $z \alpha x \sim z \alpha y$.

A5. If $x \succ y$ then $x \alpha y \succsim x \beta y \longleftrightarrow P(\alpha) \geq P(\beta)$.

A6. For $\alpha, \beta, \gamma, \delta \in E$, if $P(\delta) = P(\alpha)P(\gamma) + P(\beta)[1 - P(\gamma)]$
then $(x \alpha y) \gamma (x \beta y) \sim x \delta y$.
(consider the compound gamble. x occurs if both $\gamma, \alpha$ or if $\beta$ but not $\gamma$ occur. By the relation among probabilities, then, $P(x) = P(\delta)$. The axiom says that either implicitly or explicitly, people can compute compound probabilities, and so is a very strong assumption. Without it or an equivalent the representation theorem cannot be proved).

Sketch of Proof of the Representation Theorem. Define $u(a) = 1$, $u(b) = 0$. Let $x \in G$ . By A2 $a \succsim x \succsim b$ .
By A3, $\exists \alpha \ni x \sim a \alpha b$ Define $u(x) = P(\alpha)$ We must now show that u behaves properly :

Property (i): Let $y \sim a \beta b$ as above for x. $x \succsim y \longleftrightarrow a \alpha b \succsim a \beta b$ (by A 4) $\longleftrightarrow P(\alpha) \geq P(\beta)$ (by A 5) $\longleftrightarrow u(x) \geq u(y)$

(definition of u).
Property (ii): We have shown how to represent x,y in terms of a,b. $x\gamma y \sim (a\alpha\beta)\gamma(a\beta b)$ (by A4) $\sim a\delta b$ (by A6, where $\delta$ is such that A6 holds). Hence $u(x\gamma y) = u(a\delta b) = P(\delta)$ (by definition) $= P(\alpha)P(\gamma) + [1-P(\gamma)]P(\beta) = u(x)P(\gamma) + u(y)[1-P(\gamma)]$.

Another consequence of the representation theorem, first proved by Kemeny and Thompson and later by J. Pfanzagl (Naval Research Logistics Quarterly 6, 1959, 283-94) follows. Suppose we have gambles involving money, and $m(a\alpha b)$ is the amount of money indifferent to the gamble $a\alpha b$

> Theorem 3. If $m[(a+c)\alpha(b+c)] = m(a\alpha b) + c$ (consistency assumption) and if the utility function, u, is continuous and satisfies the expected utility hypotheis, then, either
> a. $u(x) = ax + b$, $(a>0)$ or
> b. $u(x) = a\lambda^{x} + b$, $(a>0, \lambda>1$ or $a<0, \lambda<1)$.

Note that we are forced to make the consistency assumption in experiments, since a person's wealth changes during an experiment, in his mind if not in his pocket.

## 1.2 Testing the theory : The Mosteller-Nogee Experiment.

In the first part of this experiment, subjects could either accept or reject each of a series of gambles of the form $x\alpha 5¢$ Each event was defined by a set of 5 numbers, analogous to a poker hand. Possible sequences were ranked, as in poker. In accepting the gamble, the subject was betting that the set of numbers generated by a throw of 5 dice would be better than the stated combination. The subject had before him a chart giving for each combination the probability that the dice would do better. The experiment consisted, for each event and each subject, of varying x until an indifference point was found.

Results of the experiment were as follows : The curve of probability of accepting the bet versus x was not a step function, as assumed by the von Neumann theory, but was more akin to a normal ogive; preference seemed to be variable rather than fixed. Defining the indifference point as that value of x for which the gamble was accepted half the time, Mosteller and Nogee were able to calculate a utility - of - money function for each subject. The resulting curves were qualitatively different from subject to subject.

In the second part of the experiment, subjects were offered choices between gambles. Two sets of predictions were made, one using expected utility as the criterion (based on the utility functions) and one using expected money return. When the gambles differed only a small amount in their expectations (in utilities or in money) neither prediction method succeeded. When differences were large, the utility criterion led to better predictions than did the money criterion. It is hard to say how strong is the support given by this experiment to the expected utility hypothesis.

Later work has proceeded in two directions. The von Neumann-Morgenstern theory assumes subjective values, but not subjective probabilities. In one type of work the probability assumption is replaced by something weaker. The second type of work takes seriously the observation that preference is not a step function -- that people are not algebraic. Since preference behavior often seems to be stochastic, a possible theory of preference should be also. The alternative is to develop an error theory for the algebraic model. This alternative is difficult, partly because errors depend strongly on the choices offered. For example, when choices are between pure sums of money, no error theory is needed. This argument leads to the development of a theory that is intrinsically stochastic.

## 2. Possible Forms of the Utility Function

In Theorem 3 above we stated that a consistency axiom for u, together with the expected utility axiom and the continuity of u, imply that u must be a member of a restricted class of functions. The consistency axiom has, however, been questioned, and there is some evidence against it. We shall discuss other assumptions that lead to a different restriction on the possible utility functions.

An important aspect of any scale is the set of admissible transformations of the scale, i.e. the set of transformations that do not alter the information represented by the scale. The scales that have been most studied have been interval scales (arbitrary zero and unit) and ratio scales (arbitrary unit); the corresponding sets of admissible transformations are the positive linear group and the affine group, respectively.

Theories may be divided into two classes, measurement theories and substantive theories. A measurement theory describes "unitary" relations among objects or events; it includes a set of axioms about primitive experimental operations that can be checked experimentally. (Examples are axioms about the behavior of weights on balances,and the axioms A1-6 given above about preferences for gambles). If the axioms are satisfied, then some representation in terms of a known mathematical system is possible. A substantive theory states a relationship between two measurement theories. (The measurement theories for mass and acceleration are related by Newton's second law, a substantive theory).

Suppose that we have two confirmed measurement theories, producing numerical scales, and a substantive theory relating them. Let u be the function relating the independent variable x and the dependent variable u(x). If both variables are ratio scales, and if we assume u to exist, do we know anything about it ? We do, if we are willing to accept certain assumptions about internal consistency : (1) admissible transformations of the independent variable induce only admissible transformations of the dependent variable, and (2) the form of u is independent of the choice of units for dependent and independent variables.

In particular, suppose we have two ratio scales, and a continuous non-constant u maps ratio into ratio. Then $x \to kx$ is admissible, $u \to ku$ is admissible, and so we must have u(kx) =

$K(k)\ u(x)$. But this restricts u, for let $x = 1$. Then $K(k) = \frac{u(k)}{u(1)}$. Thus $\frac{u(kx)}{u(1)} = \frac{u(k)}{u(1)} \frac{u(x)}{u(1)}$. Define $v(x) = \log \frac{u(x)}{u(1)}$.

Then $v(kx) = v(k) + v(x)$. This is a well known functional equation with $v(x) = a \log x$ as its unique continuous solutions. Thus $u(x) = \alpha x^{\beta}$ is the only possible form for u.

A utility function transforms a ratio scale (money) into an interval scale (utiles). For the first, $x \rightarrow kx$ is admissible; for the second $u \rightarrow ku + C\ (k > 0)$, is admissible. We must therefore have

$$u(kx) = K(k)u(x) + C(k),\ K(k) > 0$$

To facilitate the proof, we assume additionally that u is differentiable. (The result is also true for u continuous only). Then

$$u'(kx)k = K(k)u'(x)$$

giving $u'(kx) = \frac{K(k)}{k} u'(x)$. By the result above, this equation has the unique solution $u'(x) = \alpha x^{\beta}$, and so we get

$$u(x) = \begin{cases} ax^{b}+c, & (\beta \neq -1) \\ a \log x + c, & (\beta = -1) \end{cases}$$

The utility of money is either a power function of money with an additive constant, or a logarithmic function of money with an additive constant.

Are the functional equations acceptable ? The only way to avoid them is to multiply the ratio scale x by a constant, $\lambda$ whose units are the inverse of the units of the ratio scale; the relust is dimensionless. In this case, multiplying x by a scale factor, k, need not impose a transformation on u(x); we need only change $\lambda$ to $\lambda / K$. An example of this situation in physics is given by radioactive decay laws : time is variable, but an associated constant has the dimension time. It is a problem why certain laws of physics are expressible without forcing the quantities to be dimensionless in this way, and others are not thus expressible. If we do not assume dimensionless money as the independent variable, and we assume the existence of a utility function, then the possibilities are restricted as above. The second form is the one originally postulated by Bernoulli, but there is some weak evidence favoring the first.

## 3. A Probabilistic Utility Theory

Mosteller and Nogee found that subjects' choices were not consistent for a particular gamble; a gamble was accepted on some trials, rejected on others. On the other hand, the behavior was lawful in the sense that the probability of acceptance changed in a regular way with the expected payoff. This sort of result has been found in every behavioral study of choices between gambles. Two alternative explanations are : (1) At any moment the probability of acceptance is a step function with a single step, but past choices influence the location of the step. This can't be disproved, but it is hard to treat the combination of complicated choices and learning. (2) The behavior that we observe is not changing; the learning phase has passed. But it is intrinsically probabilistic. ("Intrinsic" is used in the sense that we do not attempt to explain the source of the probabilistic quality). The second explanation leads to theories, $\psi$ while the following is an example.

Assume a set T of alternatives, and the existence of $P_T(x)$, $x \in T$, the probability that alternative x is chosen when set T is presented. (Contrary to the evidence, we are here assuming that the order in time or space in which the elements of T are presented is irrelevant; the theory can be modified to include this, but we will not do so here). Now we assume the usual probability axioms :

$$P_T(x) \geq 0, \qquad \sum_{x \in T} P_T(x) = 1, \qquad P_T(S) = \sum_{x \in S} P_T(x).$$

Can any more be said ? There must be some interconnection between the probability measures for different sets T. For example if $T = \{x,y,z,w\}$ and $S = \{x,y,z\}$ , then we expect some relation between probabilities of choices in T and S. The simplest assumption is one of independence, and is given by the Choice Axiom :

$$\text{for } x \in S \subset T \qquad \text{if } P_T(x \mid S) = \frac{P_T(x)}{P_T(S)}$$

$$\text{exists then} \quad P_S(x) = P_T(x \mid S).$$

Thus, if we know that the choice is confined to S, then the probability of choosing x is the same as if S had been presented. We are assuming a connection between the two probability measures, $P_S$ and $P_T$. If we attempt to test the choice axiom directly, we run into order problems, and so we must either insure that our experiments include no order effects, or else generalize the theory to include order. We consider some of the consequences of the axiom, which lead to indirect tests.

Theorem 1

(i) if $P_T(x) \neq 0$
then $P_S(x) \neq 0$

(ii) if $P_T(x) = 0$ and $P_T(S) \neq 0$,
then $P_S(x) = 0$

(iii) if $P_T(y) = 0$, $y \neq 0$,
then $P_T(x) = P_{T-\{y\}}(x)$

(iv) if $P_T(y) \neq 0$ for all $y \in T$,
then $P_T(x) = P_S(x) P_T(S)$.

Theorem 2 If $P_T(y) \neq 0$ for all $y \in T$, and we define $P(x,y) = P_{\{x,y\}}(x)$, then

(i) for $x,y,z, \in T$ $P(x,y)\,P(y,z)P(z,x) = P(x,z)P(z,y)P(y,x)$

(ii) $\exists$ a ratio scale $v: T \rightarrow \text{reals}$,

$$\ni P_S(x) = \frac{v(x)}{\sum_{y \in S} v(y)}$$

We note that (i) asserts that the intransitivities $x > y > z > x$ and $x > z > y > x$ have equal probability and that it implies the condition of strong stochastic transitivity, i.e.,

$$\text{if } P(x,y) \geq \tfrac{1}{2} \text{ and } P(y,z) \geq \tfrac{1}{2} \text{ then } P(x,z) \geq \begin{cases} P(x,y) \\ P(y,z) \end{cases}$$

To specialize this theory to the utility problem, we consider the question of choices between gambles, and make use of the leverage provided by a "decomposition axiom" about such choices. The choice axiom is applied both to sets of pure outcomes and to sets of chance events. The decomposition axiom is an independence assumption (weaker than the expected utility hypothesis) that allows conclusions about a choice between gambles by decomposing it into a choice between outcomes and a choice between events. Consider the gambles $a\alpha b$, $a\beta b$. The first would be chosen over the second under the conditions $a > b$ (preference) and $\alpha > \beta$ (judged probability) or $a < b$ and $\alpha < \beta$. Let $P(a,b)$ = probability that a is preferred to b, $Q(\alpha, \beta)$ = probability that $\alpha$ is judged more likely than $\beta$, $P(a\alpha b, a\beta b)$ = probability that the first gamble is chosen over the second. Then we can state the

Decomposition Axiom

$$P(a\alpha b, a\beta b) = P(a,b)\,Q(\alpha,\beta) + P(b,a)\,Q(\beta,\alpha).$$

We mentioned above that superimposing an error theory on an algebraic utility theory was difficult because the error phenomenon is uneven; although gambles are not perfectly discriminated, pure outcomes are. The present theory produces such unevenness automatically, as shown by the theorem to follow. Assume, for theorems 3 and 4, the choice axiom for P on sets of 3 or fewer gambles, the choice axiom for Q on sets of 3 or fewer events, and the decomposition axiom. Define $\gtrsim$; $\alpha \gtrsim \beta \longleftrightarrow Q(\alpha,\beta) \geq \frac{1}{2}$. Let A be the class of pure outcomes and E the class of events, so that $G = A \times E \times A$.

<u>Theorem 3</u> If $\exists\, a, b \in A \ni P(a,b) \neq 0, \frac{1}{2},$ I then $\sim$ is an equivalence relation having at most three equivalence classes.

Since $\alpha \sim \beta \longleftrightarrow Q(\alpha,\beta) = \frac{1}{2}$ the subject thinks events in an equivalence class all have the same probability. The limitation to three classes seems contrary to our experience, and so the theorem suggests that either our axioms or the condition of the theorem must be false. If the latter, then we have the unevenness mentioned above, between <u>pure</u> outcomes we can have only perfect discrimination of preference, or indifference. The strength of this result suggests that despite their apparent plausibility, the choice and decomposition axioms are so strong as to be "almost inconsistent". If we believe that the probabilities for pure outcomes can differ from 0, $\frac{1}{2}$ 1 then the theory must be discarded, but there is little evidence for this. The next theorem has more direct experimental consequences.

<u>Theorem 4</u> Suppose a, b, c, d, $\in A \ni P(a,b) = P(c,d) = 1$ and gambles $\{a\alpha b, a\beta b, c\alpha d, c\beta d\}$ where $\alpha, \beta$ are chosen so that there is confusion between any pair. Then $P(a\alpha b, c\alpha d) = P(a\beta b, c\beta d)$.

<u>Proof</u> By the decomposition axiom, $P(a\alpha b, a\beta b) = Q(\alpha,\beta)$ and $P(c\alpha d, c\beta d) = Q(\alpha,\beta)$

and so they are equal.

By Theorem 2, there is a representation in terms of scale values, and

$$\frac{V(a\alpha b)}{V(a\alpha b)+V(a\beta b)} = \frac{V(c\alpha d)}{V(c\alpha d)+V(c\beta d)}$$

This gives $\frac{V(a\alpha b)}{V(a\beta b)} = \frac{V(c\alpha d)}{V(c\beta d)}$ and the result follows.

An experimental interpretation is as follows : Suppose we offer a choice between $a\alpha b$ and $c\alpha d$ where $a > c > d > b$, and we vary $P(\alpha)$. Then the graph of $P(a\alpha b, c\alpha d)$ versus $P(\alpha)$ consists of a series of ascending steps from 0 to 1, rather than a continuous curve. The theory does not specify the number or width of the steps, making statistical tests difficult. However, Luce and Shipley have performed such an experiment and find evidence favoring a step function over a logistic (ogival) curve. The statistic used is the number of reversals : if events are numbered i along the P ($\alpha$) axis and $N_i$ is the number of times the first gamble is accepted with the $i^{th}$ event, then $N_i - N_{i-k} \leq 0$ is called a reversal. The results favor Theorem 4, and thus the assumptions leading to it.

## 4. Utility Theories with Subjective Probabilities

Work since von Neumann and Morgenstern's has proceeded in two directions. An example of the first - probabilistic theories - was discussed above. We will now consider the second - algebraic theories incorporating subjective probabilities.

W. Edwards has discovered several results concerning utility theories with subjective probabilities. The first has to do with the type of utility measurement required by such a theory. Let us define a subjective expected utility as

$$\sum_{i=1}^{m} \phi_i u_i$$

where $u_i$ is the utility and $\phi_i$ the subjective probability, corresponding to the event i. Let us consider the relevant scale types for the utilities and probabilities. (In the von Neumann-Morgenstern theory, probabilities admit no transformations, and utilities admit positive linear transformations). Suppose to begin with that both scales admit linear transformations,

$$\phi_i' = \alpha \phi_i + \beta \ , \quad u_i' = a u_i + b$$

and that for the two sets of events, $i = 1, 2, \ldots, m$ and $i = m+1, m+2, \ldots, m+n$ the relation

$$\sum_{i=1}^{m} \phi_i u_i = \sum_{j=1}^{n} \phi_{m+j} u_{m+j}$$

holds. Let us require the property that the equality must then also hold for the transformed variables, i.e.

$$\sum_{i=1}^{m} \phi_i' u_i' = \sum_{j=1}^{n} \phi_{m+j}' u_{m+j}' .$$

The problem is what restrictions are imposed by this property on the admissible transformations. We have

$$\alpha a \sum_i \phi_i u_i + \alpha b \sum_i \phi_i + a\beta \sum_i u_i + n\beta b$$

$$= \alpha a \sum_j \phi_{m+j} u_{m+j} + \alpha b \sum_j \phi_{m+j} + a\beta \sum_j u_{m+j} + m\beta b .$$

The first terms of each member are equal by definition, and so we have

$$\alpha b\left(\sum_i \phi_i - \sum_j \phi_{m+j}\right) + a\beta\left(\sum_i u_i - \sum_j u_{m+j}\right) + (n-m)\beta b = 0$$

With an additive probability scale, the first parenthesis must be zero. The second and third parentheses can be made either positive or negative, by appropriate choices of events and their numbers. We thus have either a = b = 0 or $\beta = 0$. The condition a = 0 cannot be allowed, and so $\beta = 0$, giving ratio scale for probabilities. If the probability scale is non-additive then we also have b = 0 and so utility must also be a ratio scale. It is difficult to measure utility on a ratio scale; a natural zero is not evident. We conclude that it is difficult to keep the expected utility hypothesis when non-additive subjective probabilities are introduced.

Edwards has also shown that plausible restrictions on the relation between subjective and objective probabilities leads to another sort of difficulty. We begin by noting that subjective probability is very likely not a simple monotonic function of objective probability; the way events are described matters, as well as their objective probabilities. However, let us assume that we can group events into classes and define a subjective probability measure over each class so that a sort of monotonicity (defined by assumption 3 below) obtains within classes. If E is the universe of events, we have a mapping r : E $\longrightarrow$ integers by which the classes, $E_r$, are defined, and $E = \bigcup_r E_r$. The mapping $\Psi_r : E_r \longrightarrow [0,1]$ defines subjective probabilities for events within the $r^{th}$ class, and the mappings $\Psi_r$ induce a mapping $\Psi$ : $E \longrightarrow [0,1]$ on the universe of events. In addition we have a mapping $p: E \longrightarrow [0,1]$, the objective probability measure. We assume

(1) p is completely additive

(2) E is non-atomic : if $p(A) \neq 0, \exists \subset A \ni 0 < p(B) < p(A)$.

(3) $\Psi_r$ has the following property :
For $\varepsilon > 0$, $\exists\, \delta > 0 \ni$ if $A \in E_r$ and $p(A) < \delta$ then $\Psi_r(A) < \varepsilon$.
(That is, we can find an event A with a sufficiently small objective probability so the corresponding subjective probability is small).

(4) $\Psi$ has the following property :
If p(A) = 0 or 1,
then $\Psi$ (A) = 0 or 1 respectively.

<u>Theorem 5</u> If $\exists$ event $H \ni \Psi(H) \neq p(H)$, then for any $p$, $0 < p < 1$,
$\exists$ events $A, B \ni p(A) = p$
and $p(B) < p(A)$
but $\Psi(B) > \Psi(A) + p(1-p)\,|\Psi(H) - p(H)|$.

Thus, if the subjective and objective measures differ, then it is always possible to find a pair of events, one with a given objective probability, such that the inequality of subjective probabilities is opposite in direction from the inequality of objective probabilities. Under these conditions, then, a non-trivial subjective probability measure must be non-monotonic.

> Theorem 6 If the image of E under the mapping r is finite, then $\psi = \rho$

Thus if events are grouped into classes, each with its own subjective probability measure as above, non-trivial subjective probability measure requires an infinite collection of events. We would prefer the theory to hold for finite sets of events, corresponding to what is possible experimentally.

Edwards' results show difficulties in the subjective probability approach. With an additive measure of subjective probability, an infinite set of events is required. And even with a non-additive measure, utility must be measurable by means of a ratio scale.

Two ways of avoiding these difficulties are shown in the work of Savage, and of Davidson, Suppes and Segel. Savage does this by assuming there is no "objective probability" but only the subjective variety. Davidson et al do it by assuming a sparser set of events.

Savage thinks that the utility and probability problems lie together at the foundations of probability; without a utility measure we cannot talk about probability. He assumes a set of states of the world (events), a set of possible actions and a consequence for each state-action combination, and considers the problem of choosing among actions. His axioms state consistency assumptions about such choices, plus the existence of a sufficient richness of events. There are no axioms about probabilities. From the axioms, Savage finds that (1) there exists a mapping of the states onto the closed unit interval which satisfies the usual probability axioms, (2) there exists a mapping of the consequences onto the real numbers which behaves like a utility function (values correspond to the ordering by choice), and (3) combining the two, one action is preferred over another if its expected utility is greater. The probability measure is obtained from a single subject and is thus "subjective"; "objective probability" arises from consensus about choices for certain situations.

Davidson, Suppes and Segel use the choice situation represented by the pay-off matrix

$$\begin{array}{c|c|c|} & \text{I} & \text{II} \\ \hline \alpha & a & c \\ \hline \bar{\alpha} & b & d \\ \hline \end{array}$$

where the subject chooses the column, and then a chance event, $\alpha$ , chooses the row. They first find a particular event - one whose subjective probability is $\frac{1}{2}$ - and then hold it fixed and observe how subject's choices depend on the payoffs. The particular event, $\alpha^*$ , is chosen by using the degenerate choice situation represented by

$$\begin{array}{c|c|c|} & \text{I} & \text{II} \\ \hline \alpha^* & a & b \\ \hline \bar{\alpha}^* & b & a \\ \hline \end{array}$$

and finding an event for which the probability that the subject chooses column I is $\frac{1}{2}$; this event is defined to have subjective probability $\frac{1}{2}$. If, in the more general choice situation, column I is preferred, then we must have

$$u(a\alpha^*b) > u(c\alpha^*d)$$

And if the expected utility hypothesis is correct, then

$$u(a)\phi(\alpha^*) + u(b)\phi(\bar{\alpha}^*) > u(c)\phi(\alpha^*) + u(d)\phi(\bar{\alpha}^*).$$

Since $\phi = \frac{1}{2}$, we get

$$u(a) + u(b) > u(c) + u(d)$$

The cases of interest are those where equality holds. By assigning zero and unit utilities to two outcomes and using this method repeatedly, utilities can be assigned to a set of outcomes. The problems dealt with by Davidson et al are, first, developing an axiom system in terms of which the method is justified, and secondly, determining whether the relevant experiment can be effectively performed.